Introduction to Commutative Algebra

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Solutions to Exercises

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Contents

Notation and Terminology

"Ring" shall always mean, if not otherwise stated, *commutative ring with an identity element*. A ring omomorphism shall always, if not otherwise stated, map the identity element of the domain to the identity element of the codomain.

Part I

Exercises

Chapter 1

Rings and Ideals

Exercise 1.1. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Solution. If x is nilpotent, then $x^n = 0$ for some $n \in \mathbb{N}$. Now, using the identity

$$(1 + x)(1 - x + x^2 - \cdots + (-1)^n x^n) = 1 + (-1)^{n+1} x^{n+1}$$

we have

$$(1 + x)(1 - x + x^2 - \cdots + (-1)^{n-1} x^{n-1}) = 1 + (-1)^n x^n = 1$$

so $1 + x$ is a unit of A.

Let x be a nilpotent, and u a unit, then $uv = 1$ for some $v \in A$. The element vx is also a nilpotent, and so $1+vx$ is a unit. Finally $u+x = u+uvx = u(1+vx)$ is a unit. □

Exercise 1.2. Let A be a ring and let $A[x]$ be the ring of polynomials in an indeterminate x, with coefficients in A. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$. Prove that

1. $f(x)$ is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, a_2, \ldots, a_n$ are nilpotent.

2. $f(x)$ is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent.

3. $f(x)$ is a zero-divisor if and only if there exists $a \neq 0$ in A such that $af(x) = 0$.

4. If $f(x), g(x) \in A[x]$, then $f(x)g(x)$ is primitive if and only if $f(x)$ and $g(x)$ are primitive.

Solution.

1. (a) If a_0 is a unit and $a_1, a_2, \ldots, a_n$ are nilpotent in A, than a_0 is a unit and $a_1 x, a_2 x^2, \ldots, a_n x^n$ are nilpotent in $A[x]$, so $f(x)$ is a unit in $A[x]$ for Exercise 1.1.

(b) If $f(x)$ is a unit in $A[x]$, then there exists its inverse $g(x) = b_0 + b_1x + \cdots + b_mx^m$. The first coefficient in $f(x)g(x)$ is a_0b_0, so it must be $a_0b_0 = 1$, and a_0 is a unit. Now we show by induction on r that

$$a_n^{1+r}b_{m-r} = 0,\ r = 0, \ldots, m \tag{1.2.1}$$

Obviously $a_nb_m = 0$, since this is the coefficient of x^{n+m} in $f(x)g(x)$, so (1.2.1) holds for $r = 0$. Now suppose (1.2.1) holds for $r = 0, 1, \ldots, k$; the coefficient of $x^{n+m-(k+1)}$ in $f(x)g(x)$ is

$$a_nb_{m-k-1} + a_{n-1}b_{m-k} + \cdots + a_{n-k-1}b_m = 0$$

and multiplying this by a_n^{1+k} we get

$$a_n^{1+(k+1)}b_{m-(k+1)} + a_{n-1}a_n^{1+k}b_{m-k} + \cdots + a_{n-k-1}a_n^ka_nb_m = 0;$$

using the inductive hypotesis

$$a_n^{1+(k+1)}b_{m-(k+1)} = 0,$$

so (1.2.1) is proved. In particular $a_n^{1+m}b_0 = 0$, but b_0 is a unit, then $a_n^{1+m} = 0$, and a_n is nilpotent. Then a_nx^n is nilpotent in $A[x]$, and, being $f(x)$ a unit, for Exercise 1.1 $f(x) - a_nx^n = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ is also a unit. It follows that a_{n-1} is nilpotent, and so on.

2. (a) If $a_0, a_1, \ldots, a_n$ are nilpotent, then $a_0, a_1x, \ldots, a_nx^n$ are nilpotent in $A[x]$, and so is $f(x)$.

   (b) If $f(x)$ is nilpotent, then $(f(x))^k = 0$ for some $k \in \mathbb{N}$. The coefficient of x^{nk} in $(f(x))^k$ is a_n^k, so it must be $a_n^k = 0$, that is, a_n is nilpotent, and so is a_nx^n, and also $f(x) - a_nx^n = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$. Hence a_{n-1} is nilpotent, and so on.

3. Let $g(x) = b_0 + b_1x + \cdots + b_mx^m$ be a polynomial of lowest degree such that $f(x)g(x) = 0$. We prove by induction on r that

$$a_{n-r}g(x) = 0,\ r = 0, \ldots, n \tag{1.2.2}$$

Since $a_nb_m = 0$, $a_ng(x)$ has degree at most $m-1$, and as $a_ng(x)f(x) = 0$, it must be $a_ng(x) = 0$, so (1.2.2) is proved for $r = 0$. Now suppose (1.2.2) holds for $r = 0, 1, \ldots, k$. The coefficient of $x^{n+m-(k+1)}$ in $f(x)g(x)$ is

$$a_nb_{m-k-1} + a_{n-1}b_{m-k} + \cdots + a_{n-k-1}b_m = 0$$

whence, since the induction yields

$$a_nb_{m-k-1} = a_{n-1}b_{m-k} = \cdots = a_{n-k}b_{m-1} = 0$$

we get $a_{n-(k+1)}b_m = 0$. Again, $a_{n-(k+1)}g(x)$ has degree less than m, and since $a_{n-(k+1)}g(x)f(x) = 0$, it must be $a_{n-(k+1)}g(x) = 0$. So (1.2.2) is proved. That is, $a_ib_j = 0$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, which means that $b_hf(x) = 0$, $h = 1, \ldots, m$ holds too.

4. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in A[x]$.

(a) If $f(x)g(x)$ is primitive, then its coefficients generate A, but these coefficients are both in $(a_0, a_1, \dots, a_n)$ and in $(b_0, b_1, \dots, b_m)$, so also $f(x)$ and $g(x)$ are primitive.

(b) Suppose $f(x)$ and $g(x)$ are primitive, while $f(x)g(x)$ is not. Then its coefficient generate a proper ideal, contained in a maximal ideal M. The coefficients of $f(x)$ and $g(x)$ are not all in M, so let i_0 be such that $a_{i_0} \notin M$ while $a_i \in M$, $i = 0, \dots, i_0 - 1$ if $i_0 > 0$, and let j_0 be such that $b_{j_0} \notin M$ while $b_j \in M$, $j = 0, \dots, j_0 - 1$ if $j_0 > 1$. The coefficient of $x^{i_0+j_0}$ in $f(x)g(x)$ is

$$c = a_{i_0} b_{j_0} + \sum_{i=0}^{i_0-1} a_i b_{i_0+j_0-i} + \sum_{j=0}^{j_0-1} a_{i_0+j_0-j} b_j$$

where one of the two sums or both might not actually appear, if $i_0 = 0$ or $j_0 = 0$ or both, and of course it is understood that $a_i = 0$ if $i > n$ and $b_j = 0$ if $j > m$. In any case c and either sum that does not vanish belong to M, so it should be $a_{i_0} b_{j_0} \in M$ too, which is impossible.

□

Exercise 1.3. Generalize the results of Exercise 1.2 to a polynomial ring in several indeterminates $A[x_1, \dots, x_n]$.

Solution. Not yet available. □

Exercise 1.4. In the ring $A[x]$, the Jacobson radical is equal to the nilradical.

Solution. Since $\mathfrak{R}_N \subseteq \mathfrak{R}_J$ always holds in any ring, we only have to show that in $A[x]$ $\mathfrak{R}_J \subseteq \mathfrak{R}_N$ holds too. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathfrak{R}_J$; then $1 - xp(x) = 1 - a_0 x - \cdots - a_n x^{n+1}$ is a unit by Proposition 1.9, and by Exercise 1.2 $p(x)$ is nilpotent. □

Exercise 1.5.

Exercise 1.6. Let A be a ring in wich every ideal not contained in the nilradical contains a non-zero idempotent (that is, an element e such that $e^2 = e \neq 0$). Then the nilradical and the Jacobson radical are equal.

Solution. As in Exercise 1.4 we need only show that $\mathfrak{R}_J \subseteq \mathfrak{R}_N$. Let $x \notin \mathfrak{R}_N$. Then $(x) \nsubseteq \mathfrak{R}_N$, so there is an element $e \in (x)$ such that $e^2 = e \neq 0$. Let $e = xt$, then $x^2 t^2 = xt$, $xt(1 - xt) = 0$, which implies that $1 - xt$ is not a unit, so by Proposition 1.9 $x \notin \mathfrak{R}_J$. □

Exercise 1.7. Let A be a ring in wich every element x satisfies $x^n = x$ for some $n \in \mathbb{N}$, $n > 1$, depending on x. Then every prime ideal in A is maximal.

Solution: first way. Let $\mathfrak{p}$ be a prime ideal in A. We shall prove that $A/\mathfrak{p}$ is a field. Let $< x >= (x + \mathfrak{p}) \neq 0$ in $A/\mathfrak{p}$. We have $x^n = x$ for some $n \in \mathbb{N}$, $n > 1$, then $< x >^n = < x >$ in $A/\mathfrak{p}$, so $< x >^{n-1} = 1$, and also $< x >^{n-2} < x >= 1$, since $n - 2 \geq 0$, thus $< x >$ is a unit in $A/\mathfrak{p}$. □

Solution: second way. Let $\mathfrak{p}$ be a prime ideal in A, $x \notin \mathfrak{p}$. We shall prove that $(x) + \mathfrak{p} = A$, hence $\mathfrak{p}$ is maximal. We have $x^n = x$ for some $n \in \mathbb{N}$, $n > 1$, so $x(x^{n-1} - 1) = 0$, then $x(x^{n-1} - 1) \in \mathfrak{p}$ and also $x^{n-1} - 1 \in \mathfrak{p}$, be $x^{n-1} - 1 = p$. So $1 = x^{n-1} - p$ with $x^{n-1} \in (x)$ since $n - 1 \geq 1$, and $-p \in \mathfrak{p}$. □

Exercise 1.8. If A is a ring, $A \neq 0$, the set of prime ideals of A has minimal elements with respect to inclusion.

Solution. We need Zorn's lemma. Call Σ the set of all prime ideals of A. Let $\{\mathfrak{a}_\lambda\}_{\lambda \in \Lambda}$ be a chain in Σ. Then

$$\mathfrak{a} = \bigcap_{\lambda \in \Lambda} \mathfrak{a}_\lambda$$

is an ideal. We show that $\mathfrak{a}$ is prime, so that any chain in Σ has a lower bound in Σ.

Let $x, y \in A$ such that $xy \in \mathfrak{a}$. If $x \notin \mathfrak{a}_\alpha$ for some $\alpha \in \Lambda$ and $y \notin \mathfrak{a}_\beta$ for some $\beta \in \Lambda$, then either $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ or $\mathfrak{a}_\beta \subseteq \mathfrak{a}_\alpha$; in the former case $y \notin \mathfrak{a}_\alpha$ also holds, in the latter case $x \notin \mathfrak{a}_\beta$: both cases yield a contradiction, since $xy \in \mathfrak{a}_\alpha$ and $xy \in \mathfrak{a}_\beta$, and both $\mathfrak{a}_\alpha$ and $\mathfrak{a}_\beta$ are prime ideals. So either $\forall \lambda \in \Lambda\, x \in \mathfrak{a}_\lambda$ and $x \in \mathfrak{a}$ or $\forall \lambda \in \Lambda\, y \in \mathfrak{a}_\lambda$ and $y \in \mathfrak{a}$, which proves that $\mathfrak{a}$ is a prime ideal.

Thus Σ has minimal elements with respect to inclusion. □

Exercise 1.9. If A is a ring, and $\mathfrak{a}$ is a proper ideal of A, then $\mathfrak{a} = r(\mathfrak{a})$ if and only if $\mathfrak{a}$ is an intersection of prime ideals.

Solution. If $r(\mathfrak{a}) = \mathfrak{a}$, then by Proposition 1.14, $\mathfrak{a}$ is an intersection of prime ideals. If $\mathfrak{a}$ is an intersection of prime ideals, then it is the intersection of all the prime ideals which contain it, so, again by Proposition 1.14, $\mathfrak{a} = r(\mathfrak{a})$. □

Exercise 1.10. Let A be a ring, $\mathfrak{R}_N$ its nilradical. The following facts are equivalent:

i) A has exactly one prime ideal;

ii) every element of A is either a unit or a nilpotent;

iii) $A/\mathfrak{R}_N$ is a field.

Solution.

- *i)* $\Rightarrow$ *ii)* Let $\mathfrak{p}$ be the only prime ideal of A. Then $\mathfrak{R}_N = \mathfrak{p}$, so for each $a \in A$: if $a \in \mathfrak{p}$, then a is nilpotent, if $a \notin \mathfrak{p}$, then a does not belong to any prime ideal of A, thus it is a unit.

- *ii)* $\Rightarrow$ *iii)* If $a \notin \mathfrak{R}_N$, then a is a unit, so $\mathfrak{R}_N$ is maximal and $A/\mathfrak{R}_N$ is a field.

- *iii)* $\Rightarrow$ *i)* If $A/\mathfrak{R}_N$ is a field, then $\mathfrak{R}_N$ is maximal, so it is the only prime ideal.

$\square$

Exercise 1.11. If A is a boolean ring, then

1. $2x = 0$ for all $x \in A$;
2. every prime ideal $\mathfrak{p}$ is maximal and $A/\mathfrak{p}$ has two elements;
3. every finitely generated ideal of A is principal.

Solution.

1. $1+1 = (1+1)^2 = (1+1)(1+1) = 1+1+1+1$, hence $2 = 1+1 = 0$ and $2x = 0$ for all $x \in A$;
2. let $\mathfrak{p}$ be a prime ideal of A, and $x \notin \mathfrak{p}$; then $x(x-1) = x^2 - x = x - x = 0 \in \mathfrak{p}$, so $y = x - 1 \in \mathfrak{p}$ and $1 = x - y$; this means that x and $\mathfrak{p}$ generate A, thus $\mathfrak{p}$ is maximal, and that $x + \mathfrak{p}$ and $1 + \mathfrak{p}$ are the same element of $A/\mathfrak{p}$, so $A/\mathfrak{p}$ has two elements.
3. Let $I = (a_1, \cdots, a_n)$ be a finitely generated ideal of A. We show by induction on n that I is principal. If $n = 1$, it is true. Let $J = (a_1, \cdots, a_{n+1})$. The induction yields that $(a_1, \cdots, a_n)$ is principal, so $(a_1, \cdots, a_n) = (a)$ and $J = (a_1) + (a)$. Now, any two elements x, y of A are multiple of $x + y + xy$: $x(x + y + xy) = x^2 + xy + x^2y = x + xy + xy = x + 2xy = x$, and $y(x + y + xy) = yx + y^2 + xy^2 = xy + y + xy = y + 2xy = y$, so $J = (a_1 + a + a_1a)$.

NB It is interesting to show explicitly which is the element that generates the ideal $I = (a_1, \cdots, a_n)$. In fact, the elements $a_1, \cdots, a_n$, are all multiple of

$$a = \sum_{k=1}^{n} \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1} a_{i_2} \cdots a_{i_k},$$

since $a_i = a_i a$, $i = 1, 2, \ldots, n$. To show this, observe that$a_i a_{i_1} a_{i_2} \cdots a_{i_k} = a_{i_1} a_{i_2} \cdots a_{i_k}$ if and only if $a_i = a_{i_h}$ for some h, that is, if and only if a_i already appears in the product $a_{i_1} a_{i_2} \cdots a_{i_k}$. Now, there are $\binom{n-1}{k-1}$ products of k factors $a_{i_1} a_{i_2} \cdots a_{i_k}$ in which a_i does appear, and these products does not change when multyplied by a_i, so in the expression

$$a_i \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1} a_{i_2} \cdots a_{i_k}, \tag{1.11.1}$$

there are all the $\binom{n-1}{k-1}$ products of k elements which contain a_i, while the other products have $k+1$ elements. On the other hand, there are $\binom{n}{k-1} - \binom{n-1}{k-2}$ products with $k-1$ elements $a_{i_1} a_{i_2} \cdots a_{i_{k-1}}$ which do not contain a_i, so in the expression

$$a_i \sum_{i_1 < i_2 < \cdots < i_{k-1}} a_{i_1} a_{i_2} \cdots a_{i_{k-1}}, \tag{1.11.2}$$

there are $\binom{n}{k-1} - \binom{n-1}{k-2}$ products with k elements $a_{i_1}a_{i_2}\cdots a_{i_k}$, while the others have $k-1$ elements. But $\binom{n}{k-1} - \binom{n-1}{k-2} = \binom{n-1}{k-1}$, so these products are *all* the products with k elements which contain a_i.
Conclusion: in the expression of

$$a_i a = a_i \sum_{k=1}^{n} \sum_{i_1<i_2<\cdots<i_{k-1}} a_{i_1}a_{i_2}\cdots a_{i_{k-1}}, \tag{1.11.3}$$

for each $k > 2$, for each product in the inner sum which contains a_i, and so does not change when multyplied by it, there is a product in the inner sum for $k-1$ which is the same, so the only product which survives is the one for $k = 1$ which contains (and actually is equal to) a_i, multyplied by a_i, and that is, in fact, a_i. □

Exercise 1.12. A local ring contains no idempotent exept 0 and 1.

Solution. Let A be a local ring, $\mathfrak{m}$ its maximal ideal, and a an idempotent element of A. If $a \in \mathfrak{m}$, $1-x$ is a unit, since $\mathfrak{R}_J = \mathfrak{m}$, so from $(1-x)(1+x) = 1-x^2 = 1-x$ follows $1+x = 1$ and $x = 0$. If $a \notin \mathfrak{m}$, a is a unit, and from $x^2 = x$ follows $x^2x^{-1} = xx^{-1}$, $x = 1$. □

Exercise 1.13.

Exercise 1.14. In a ring A the set Σ of all ideals in which every element is a zero-divisor has maximal elements, and every maximal element of Σ is a prime ideal. Hence the set of zero-divisors of A is a union of prime ideals,

Solution. We need Zorn's Lemma. Let S be a chain of ideals of Σ, and

$$\mathfrak{u} = \bigcup_{\mathfrak{a}\in S} \mathfrak{a}.$$

The set $\mathfrak{u}$ is an ideal: this is an easy check. Also, $\mathfrak{u} \in \Sigma$, that is, every element of $\mathfrak{u}$ is a zero-divisor: this is also an easy check. So $\mathfrak{u}$ is an upper bound in Σ for the chain S. By Zorn's Lemma, Σ has maximal elements.
Now, let $\mathfrak{m}$ be a maximal element of Σ, and $x, y \in A$ such that $xy \in \mathfrak{m}$. We shall prove that both $x \notin \mathfrak{m}$ and $y \notin \mathfrak{m}$ yields a contradiction. If that is the case, then $\mathfrak{m} \subset \mathfrak{m} + (x)$ and $\mathfrak{m} \subset \mathfrak{m} + (y)$, which means, being $\mathfrak{m}$ a maximal element of Σ, that $m + (x) \notin \Sigma$, and $m + (y) \notin \Sigma$, that is, there are elements m', $m'' \in \mathfrak{m}$ and a, $b \in A$ such that $m' + ax$ and $m'' + by$ are not zero-divisors. But

$$m = (m' + ax)(m'' + by) = m'm'' + m'by + m''ax + abxy \in \mathfrak{m}$$

so m is a zero divisor, which is impossible by Proposition 1.1 of part II. □

Exercise 1.15. Let A be a ring, and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ be the set of all prime ideals of A which contain E. Then

1. if $\mathfrak{a} = (E)$, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$;
2. $V(0) = X$, $V(1) = \emptyset$;

3. if $\{E_i\}_{i\in I}$ is any family of subset of A, then
$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V\left(E_i\right);$$

4. $V\left(\mathfrak{a}\cap\mathfrak{b}\right) = V\left(\mathfrak{a}\mathfrak{b}\right) = V(\mathfrak{a})\cup V(\mathfrak{b})$.

Solution.

1.
   - Of course $V(\mathfrak{a}) \subseteq V(E)$ since $E \subseteq \mathfrak{a}$. On the other hand, $\mathfrak{a}$ is contained in every ideal which contains E, so $V(E) \subseteq V(\mathfrak{a})$.
   - Of course $V(r(\mathfrak{a})) \subseteq V(\mathfrak{a})$ since $\mathfrak{a} \subseteq r(\mathfrak{a})$. But $r(\mathfrak{a})$ is the intersection of all the prime ideals which contain $\mathfrak{a}$, so if $\mathfrak{p}$ is a prime ideal and $\mathfrak{a} \subseteq \mathfrak{p}$, then also $r(\mathfrak{a}) \subseteq \mathfrak{p}$, hence $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$.
2.
   - $\forall\mathfrak{p} \in X\; 0 \in \mathfrak{p}$, hence $X \subseteq V(0)$ and $X = V(0)$.
   - $\forall\mathfrak{p} \in X\; 1 \notin \mathfrak{p}$, hence $V(0) = \emptyset$.
3. We have
$$\mathfrak{p} \in V\left(\bigcup_{i\in I} E_i\right) \iff \mathfrak{p} \supseteq \bigcup_{i\in I} E_i \iff \forall i \in I\; \mathfrak{p} \supseteq E_i$$
$$\iff \forall i \in I\; \mathfrak{p} \in V(E_i) \iff \mathfrak{p} \in \bigcap_{i\in I} V\left(E_i\right).$$

- Of course $V\left(\mathfrak{a}\cap\mathfrak{b}\right) \subseteq V\left(\mathfrak{a}\mathfrak{b}\right)$, since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a}\cap\mathfrak{b}$. On the other hand, if $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$ and $x \in \mathfrak{a}\cap\mathfrak{b}$, then $x \in \mathfrak{a}$ and $x \in \mathfrak{b}$, so $x^2 \in \mathfrak{a}\mathfrak{b}$, hence $x^2 \in \mathfrak{p}$ and $x \in \mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal; then $\mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{p}$ and $\mathfrak{p} \in V(\mathfrak{a}\cap\mathfrak{b})$; hence $V\left(\mathfrak{a}\mathfrak{b}\right) \subseteq V\left(\mathfrak{a}\cap\mathfrak{b}\right)$.
- $\mathfrak{p} \in V(\mathfrak{a}\cap\mathfrak{b}) \iff \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{p} \iff \mathfrak{a} \subseteq \mathfrak{p} \vee \mathfrak{b} \subseteq \mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal, so $\mathfrak{p} \in V(\mathfrak{a}\cap\mathfrak{b}) \iff \mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$.

□

Exercise 1.16.

Exercise 1.17. Let A be a ring, $X = \mathrm{Spec}(A)$ and for each $f \in A$ let X_f be the complement of $V((f))$ in X. The sets X_f are a basis of the open sets for the Zariski topology of X, and

1. $X_f \cap X_g = X_{fg}$
2. $X_f = \emptyset \iff f \in \mathfrak{R}_N$
3. $X_f = X \iff f$ is a unit
4. $X_f = X_g \iff r((f)) = r((g))$
5. X is quasi-compact (that is, every open covering of X has a finite subcovering)
6. each X_f is quasi-compact

7. an open subset of X is quasi-compact if and only if it is a finite union of sets X_f

Solution. If U is an open set in X, then $U = V(F)'$ for some $F \subseteq A$, so

$$U = V\left(\bigcup_{f\in F}\{f\}\right)' = \left(\bigcap_{f\in F} V(\{f\})\right)' = \bigcup_{f\in F} V((f))' = \bigcup_{f\in F} X_f$$

hence the sets X_f are a basis for the Zariski topology of X.
Furthermore

1. $\mathfrak{p} \in X_f \cap X_g \iff \mathfrak{p} \notin V((f)) \cup V((g))$ but $V((f)) \cup V((g)) = V((f)(g)) = V((fg))$ then $\mathfrak{p} \in X_f \cap X_g \iff \mathfrak{p} \in X_{fg}$.

2. $X_f = \emptyset \iff \forall \mathfrak{p} \in X\, \mathfrak{p} \in V((f)) \iff \forall \mathfrak{p} \in X\, (f) \subseteq \mathfrak{p} \iff f \in \mathfrak{R}_N$

3. $X_f = X \iff \forall \mathfrak{p} \in X\, \mathfrak{p} \notin V((f)) \iff \forall \mathfrak{p} \in X\, (f) \nsubseteq \mathfrak{p} \iff \forall \mathfrak{p} \in X f \notin \mathfrak{p} \iff f$ is a unit

4. $X_f \subseteq X_g \iff V((g)) \subseteq V((f)) \iff \forall \mathfrak{p}\, \mathfrak{p} \in V((g)) \Rightarrow \mathfrak{p} \in V((f)) \iff \forall \mathfrak{p} \in X\, (g) \subseteq \mathfrak{p} \Rightarrow (f) \subseteq \mathfrak{p} \iff (f) \subseteq r((g)) \iff r((f)) \subseteq r((g))$

5. To show that X is quasi-compact, it is enough to show that any covering of X with sets X_f has a finite subcovering. So suppose that for some $F \subseteq A$:

$$X = \bigcup_{f\in F} X_f.$$

Then

$$X = \bigcup_{f\in F} V(\{f\})' = \left(\bigcap_{f\in F} V(\{f\})\right)' = \left(V\left(\bigcup_{f\in F}\{f\}\right)\right)' = \\ = V(F)'$$

so $\forall \mathfrak{p} \in X\, F \nsubseteq \mathfrak{p}$, which means that F generates A, so there are elements $f_1, f_2, \ldots, f_n$ in F and elements $a_1, a_2, \ldots, a_n$ in A such that $1 = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$, hence

$$\forall \mathfrak{p} \in X\, \mathfrak{p} \nsupseteq (f_1, f_2, \ldots, f_n) = (f_1) + (f_2) + \cdots + (f_n)$$

and

$$\forall \mathfrak{p} \in X\, \mathfrak{p} \in V\left(\sum_{i=1}^{n}(f_i)\right)' = V\left(\bigcup_{i=1}^{n}(f_i)\right)' = \left(\bigcap_{i=1}^{n} V((f_i))\right)' \\ = \bigcup_{i=1}^{n} V((f_i))' = \bigcup_{i=1}^{n} X_{f_i}.$$

6. If for some $G \subseteq A$ it is

$$X_f \subseteq \bigcup_{g \in G} X_g$$

that means $\forall \mathfrak{p} \in X \, \mathfrak{p} \supseteq G \Rightarrow \mathfrak{p} \supseteq (f)$, hence $(f) \subseteq r((G))$ and for some $n \in \mathbb{N}$ we have $f^n \in (G)$, then for some elements $g_1, g_2, \ldots, g_n$ of G and $a_1, a_2, \ldots, a_n$ in A $f^n = a_1 g_1 + a_2 g_2 + \cdots + a_n g_n$. So $\mathfrak{p} \in V(g_1, g_2, \ldots, g_n) \Rightarrow f^n \in \mathfrak{p} \Rightarrow f \in \mathfrak{p}$, since $\mathfrak{p}$ is prime, and $\mathfrak{p} \in V((f))$, that is $V(g_1, g_2, \ldots, g_n) \subseteq V((f))$. But

$$V(g_1, g_2, \ldots, g_n) = \left(\bigcup_{i=1}^{n} X_{g_i} \right)'$$

so

$$X_f \subseteq \bigcup_{i=1}^{n} X_{g_i}.$$

7. If Y is an open subset of X, then

$$Y = \bigcup_{f \in F} X_f$$

for some $F \subseteq A$, and if Y is compact there is a finite subcovering of Y:

$$Y \subseteq \bigcup_{i=1}^{n} X_{f_i}, \quad f_i \in F, \, i = 1, 2, \ldots, n;$$

but we have also

$$\bigcup_{i=1}^{n} X_{f_i} \subseteq Y$$

hence

$$Y = \bigcup_{i=1}^{n} X_{f_i}$$

and so a compact open subset of X is a finite union of sets X_f.
On the other hend, the subsets X_f are compact, and a finite union of compact subsets is always compact. □

Exercise 1.18. Let A be a ring, $X = \mathrm{Spec}(A)$, $x \in X$.

1. The set $\{x\}$ is closed in X if and only if x is a maximal ideal of A.
2. $\overline{\{x\}} = V(x)$
3. $y \in \overline{\{x\}} \iff x \subseteq y$
4. X is a T_0 space

Solution.

1. If x is a maximal ideal of A, then x is the only element of X which contains x, so $V(\{x\}) = \{x\}$ and $\{x\}$ is closed in X. On the other hand, if $\{x\}$ is closed in X, it is $\{x\} = V(E)$ for some $E \subseteq A$, which means that x is the only prime ideal that contains E, so x is maximal.

2. Of course $\overline{\{x\}} \subseteq V(x)$, since $V(x)$ is closed and $x \in V(x)$ so $\{x\} \subseteq V(x)$. On the other hand, if C is a closed subset of X and $\{x\} \subseteq C$, then $C = V(E)$ for some $E \subseteq A$ and $\{x\} \subseteq V(E)$, so $x \in V(E)$, $E \subseteq x$, and $V(x) \subseteq V(E) = C$: $V(x)$ is a subset of any closed subset of X which contains $\{x\}$, so $V(x) \subseteq \overline{\{x\}}$.

3. Since $\overline{\{x\}} = V(x)$, $y \in \overline{\{x\}} \iff y \in V(x) \iff x \subseteq y$.

4. Let $x, y \in X$, $x \neq y$. Then either $x \nsubseteq y$ or $y \nsubseteq x$. In the former case $y \notin V(x)$, while $x \in V(x)$, so $y \in V(x)'$, $x \notin V(x)'$; in the latter case $x \in V(y)'$, $y \notin V(y)'$.

□

Exercise 1.19. Spec(A) is irreducible if and only if $\mathfrak{R}_N$ is a prime ideal.

Solution. Let $X = \text{Spec}(A)$.

- Let $\mathfrak{R}_N$ be a prime ideal, U_1, U_2 two non-empty open subsets of X. Then $U_1 = V_1', U_2 = V_2'$ where both V_1, V_2 are proper closed subsetes of X, so $\mathfrak{R}_N \notin V_1$ and $\mathfrak{R}_N \notin V_2$ and also $\mathfrak{R}_N \notin V_1 \cup V_2$, hence $V_1 \cup V_2$ is a proper closed subset of x, and $U_1 \cap U_2 = (V_1 \cup V_2)'$ is not empty.

- If $\mathfrak{R}_N$ is not a prime ideal, there are two elements $f, g \in A$ such that $fg \in \mathfrak{R}_N$, $x \notin \mathfrak{R}_N, y \notin \mathfrak{R}_N$. Then X_f and X_g are both non-empty and $X_f \cap X_g = (V((f)) \cup V((g)))' = (V((fg)))' = X' = \emptyset$.

□

Exercise 1.20. Let A be a ring, $X = \text{Spec}(A)$. The irreducible components of X are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of A.

Solution. Let $\mathfrak{p} \in X$. To prove that $V(\mathfrak{p})$ is irreducible, it is enough to prove that every pair of non-empty open sets of $V(\mathfrak{p})$ of the form $X_f \cap V(\mathfrak{p})$ have non-empty intersection. So let X_f, X_g be such that both $X_f \cap V(\mathfrak{p}) \neq \emptyset$ and $X_g \cap V(\mathfrak{p}) \neq \emptyset$. That means $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, so $fg \notin \mathfrak{p}$ and $X_{fg} \cap \mathfrak{p} \neq \emptyset$. But $X_{fg} = X_f \cap X_g$, hence $X_f \cap X_g \cap \mathfrak{p} = (X_f \cap \mathfrak{p}) \cap (X_g \cap \mathfrak{p}) \neq \emptyset$.
Of course, if $\mathfrak{p}$ is a minimal prime ideal of A, then $V(\mathfrak{p})$ is a maximal irreducible subspace, that is, an irreducible component of X.
Now we have to prove that the sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is minimal prime ideal of A, are the only irreducible components of X. So let C be an irreducible component of X. Since C is closed, it is $C = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A. If $\mathfrak{a}$ is not a prime ideal, there are two elements f, g of A such that $f \notin \mathfrak{a}$ and $g \notin \mathfrak{a}$ but $fg \in \mathfrak{a}$, then $X_f \cap V(\mathfrak{a}) \neq \emptyset$, $X_g \cap V(\mathfrak{a}) \neq \emptyset$, and $(X_f \cap \mathfrak{p}) \cap (X_g \cap \mathfrak{p}) = \emptyset$. □

Exercise 1.21. Let $\phi : A \to B$ be a ring homomorphism, $X = \text{Spec}(A), Y = \text{Spec}(B)$.

1. If $f \in A$ then ${\phi^*}^{-1}(X_f) = Y_{\phi(f)}$, hence ϕ^* is continuous.
2. If $\mathfrak{a}$ is an ideal of A then ${\phi^*}^{-1}(V(\mathfrak{a})) = V((\phi(\mathfrak{a})))$
3. If $\mathfrak{b}$ is an ideal of B then $\overline{\phi^*(V(\mathfrak{b}))} = V(\phi^{-1}(\mathfrak{b}))$
4. If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto $V(\mathrm{Ker}(\phi))$
5. $\phi^*(Y)$ is dense in X if and only if $\mathrm{Ker}(\phi) \subseteq \mathfrak{R}_N$. In particular, if ϕ is injective then $\phi^*(Y)$ is dense in X
6. If $\psi : B \to C$ is another ring homomorphism, then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$
7. Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$, and let K be the field of fraction of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\phi : A \to B$ by $\phi(x) = (x + \mathfrak{p}, x)$. Then ϕ^* is bijective but not a homeomorphism.

Solution.

1. $\mathfrak{q} \in {\phi^*}^{-1}(X_f) \iff \phi^*(\mathfrak{q}) \in X_f \iff f \notin \phi^*(\mathfrak{q}) \iff \phi(f) \notin \mathfrak{q} \iff \mathfrak{q} \in Y_{\phi(f)}$
2.
   - $\mathfrak{q} \in {\phi^*}^{-1}(V(\mathfrak{a})) \Rightarrow \phi^*(\mathfrak{q}) \in V(\mathfrak{a}) \Rightarrow \mathfrak{a} \subseteq \phi^*(\mathfrak{q}) \Rightarrow \mathfrak{a} \subseteq \phi^{-1}(\mathfrak{q}) \Rightarrow \phi(\mathfrak{a}) \subseteq \phi(\phi^{-1}(\mathfrak{q})) \subseteq \mathfrak{q} \Rightarrow \mathfrak{q} \in V(\phi(\mathfrak{a})) = V((\phi(\mathfrak{a})))$
   - $\mathfrak{q} \in V((\phi(\mathfrak{a}))) \Rightarrow \phi(\mathfrak{a}) \subseteq \mathfrak{q} \Rightarrow \phi^{-1}(\phi(\mathfrak{a})) \subseteq \phi^{-1}(\mathfrak{q})$ but $\mathfrak{a} \subseteq \phi^{-1}(\phi(\mathfrak{a}))$ so $\mathfrak{a} \subseteq \phi^{-1}(\mathfrak{q}) = \phi^*(\mathfrak{q}) \Rightarrow \phi^*(\mathfrak{q}) \in V(\mathfrak{a}) \Rightarrow \mathfrak{q} \in {\phi^*}^{-1}(V(\mathfrak{a}))$
3. By Proposition (1.2) of Part II

$$\overline{\phi^*(V(\mathfrak{b}))} = V\left(\bigcap_{\mathfrak{q}\in\phi^*(V(\mathfrak{b}))} \mathfrak{q}\right).$$

Now, $\mathfrak{q} \in \phi^*(V(\mathfrak{b})) \iff \exists \mathfrak{r} : \mathfrak{r} \in V(\mathfrak{b}) \wedge \mathfrak{q} = \phi^*(\mathfrak{r})$, then

$$\begin{aligned}\overline{\phi^*(V(\mathfrak{b}))} = V\left(\bigcap_{\mathfrak{r}\in V(\mathfrak{b})} \phi^*(\mathfrak{r})\right) = V\left(\bigcap_{\mathfrak{r}\in V(\mathfrak{b})} \phi^{-1}(\mathfrak{r})\right) = \\ = V\left(\phi^{-1}\left(\bigcap_{\mathfrak{r}\in V(\mathfrak{b})} (\mathfrak{r})\right)\right) = V\left(\phi^{-1}(r(\mathfrak{b}))\right) = \\ = V\left(r(\phi^{-1}(\mathfrak{b}))\right) = V\left(\phi^{-1}(\mathfrak{b})\right).\end{aligned}$$

□

Part II

Notes

Chapter 1

Rings and Ideals

Proposition 1.1. *If xy is a zero-divisor, then at least one of the two elements x, y is a zero-divisor.*

Proof. If xy is a zero-divisor, then there is an element $t \neq 0$ such that $xyt = 0$. If $yt = 0$, then y is a zero-divisor; if $yt \neq 0$, then x is a zero-divisor. $\square$

Proposition 1.2. *Let A be a ring, $X = \operatorname{Spec}(A)$. If $Y \subseteq X$, then*

$$\overline{Y} = V\left(\bigcap_{\mathfrak{p}\in Y} \mathfrak{p}\right).$$

Proof. Let

$$\widetilde{Y} = V\left(\bigcap_{\mathfrak{p}\in Y} \mathfrak{p}\right).$$

Of course $\overline{Y} \subseteq \widetilde{Y}$, since $\widetilde{Y}$ is closed and $Y \subseteq \widetilde{Y}$. On the other hand, if C is a closed subset of X and $Y \subseteq C$, then $C = V(E)$ for some $E \subseteq A$, and $\forall \mathfrak{p}\ \mathfrak{p} \in Y \Rightarrow \mathfrak{p} \in V(E) \Rightarrow E \subseteq \mathfrak{p}$ which means that

$$E \subseteq \bigcap_{\mathfrak{p}\in Y} \mathfrak{p}$$

and so $\widetilde{Y} \subseteq V(E) = C$. Thus $\widetilde{Y} \subseteq C$ for every closed subset C of X that contains Y, so $\widetilde{Y} \subseteq \overline{Y}$. $\square$